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THE EQUIVALENCE OF WEAK SOLUTIONS AND ENTROPY SOLUTIONS AND APPLICATIONS

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1. INTRODUCTION

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, with Lipschitz boundary $\partial\Omega$ and let $T > 0$. Given an $f \in L^1(Q)$ (where $Q = (0,T) \times \Omega$) and a $g_0 \in L^1(\Omega)$ consider the initial-boundary value problem

$$(E) \quad \begin{cases} \frac{\partial g(u)}{\partial t} - \triangle b(u) + \operatorname{div} \phi(u) = f & \text{in } Q, \\ b(u) = 0 & \text{on } (0,T) \times \partial\Omega, \\ g(u)(0,\cdot) = g_0 & \text{in } \Omega, \end{cases}$$

where

(H1) $g, b : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing functions satisfying $g(0) = b(0) = 0$, and $\phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous function satisfying $\phi(0) = 0$.

There exists a vast literature on problems of this type. A number of different notions of solutions for these problems has been introduced, and the existence and uniqueness of such solutions has been studied by many authors (cf. e.g., [1-9, 12-16]).

Throughout this paper we always assume that $f \in L^2(0,T;H^{-1}(\Omega)) \cap L^1(Q)$, $g_0 \in L^1(\Omega)$ and $g_0(x) \in R(g)$ for a.e. $x \in \Omega$. Let us first recall from [8] the definition of weak solution of (E).

Definition 1.1. *A weak solution of (E) is a measurable function u satisfying*

$$\begin{aligned} & g(u) \in L^1(Q), \quad \frac{\partial g(u)}{\partial t} \in L^2(0,T;H^{-1}(\Omega)), \\ & b(u) \in L^2(0,T;H_0^1(\Omega)), \quad \phi(u) \in (L^2(\Omega))^N, \\ & \frac{\partial g(u)}{\partial t} - \triangle b(u) + div\, \phi(u) = f \qquad in\, \mathcal{D}'(Q), \\ & g(u(0,x)) = g_0(x) \qquad a.e.\ in\, \Omega. \end{aligned}$$

For the existence of a weak solution we refer to [1] and [4]. Due to the possible degeneracy of b and g, in general, one can not expect that weak solutions of (E) are unique. To prove the uniqness Carrillo [8] introduced the following notion of entropy solution for (E) as in Kružhkov [14] and obtained the L^1-contraction and uniqueness of entropy solutions of (E).

In what follows, H denotes the multi-valued function defined by $H(r) = 0$ if $r < 0$, $H(0) = [0,1]$, $H(r) = 1$ if $r > 0$ and we denote by $H_j, j = 0,1$, its single-valued section which takes the value j at $r = 0$.

Definition 1.2. *An entropy solution of (E) is a weak solution u satisfying*

$$\begin{aligned}&\int_Q H_0(u-s)\{\nabla b(u)\cdot\nabla\xi - (\phi(u)-\phi(s))\cdot\nabla\xi \\ &\quad -((g(u)-g(s))\xi_t - f\xi\}dtdx \\ &\quad \leq \int_\Omega (g_0 - g(s))^+\xi(0)dx\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}&\int_Q H_0(-s-u)\{\nabla b(u)\cdot\nabla\xi - (\phi(u)-\phi(-s))\cdot\nabla\xi \\ &\quad -((g(u)-g(-s))\xi_t - f\xi\}dtdx \\ &\quad \geq -\int_\Omega (g_0 - g(-s))^-\xi(0)dx\end{aligned} \tag{1.2}$$

for any $(s,\xi) \in \mathbb{R}^+ \times C_0^\infty([0,T)\times\bar{\Omega})^+$ and for any $(s,\xi) \in \mathbb{R}\times C_0^\infty([0,T)\times\Omega)^+$. Here, $r^+ = \max\{r,0\}$, $r^- = -\min\{r,0\}$, $\mathbb{R}^+ = [0,\infty)$ and X^+ denotes all nonnegative functions which belong to X, where $X = C_0^\infty([0,T)\times\bar{\Omega})$ or $X = C_0^\infty([0,T)\times\Omega)$.

On the other hand, Brézis and Crandall [6] has proved the uniqueness of weak solutions of the equation of porous medium type that is a special case of (E) where $g(r) = r$ and $\phi(r) = 0$ for all $r \in \mathbb{R}$. Their idea of the proof is to apply $(I - \varepsilon\triangle)^{-1}$ to the difference of equations. This technique requires the use of the existence theory (nonlinear semigroup theory) to get the L^1-contraction for weak solutions. However, to our knowlege, it is unknown that a weak solution of the porous medium equation is indeed an entropy solution in the sense of the above definition. Thus it is worthwhile to ask whether or not a weak solution of (E) indeed becomes an entropy solution of (E). If this is true, then we can prove the L^1-contraction and uniqueness of weak solutions without the use of existence theorem.

The purpose of this paper is to investigate the equivalence of weak solutions and entropy solutions under the additional assumption on ϕ.

2. The main result and Examples

We assume the following additional condition.

(H2) There exist functions $\phi^{(1)}, \phi^{(2)} \in C(\mathbb{R},\mathbb{R}^N)$ and constants $C, r_0 > 0$ such that

$$\phi^{(1)}(0) = 0, \quad \phi^{(2)}(b(r))g(r) = 0 \quad \text{if } b(r) = 0, \tag{2.1}$$

$$\phi(r) = \phi^{(1)}(b(r)) + \phi^{(2)}(b(r))g(r), \quad r \in \mathbb{R}, \tag{2.2}$$

$$|\phi(r)| \leq Cb(r)^2 \quad \text{for } |r| \geq r_0. \tag{2.3}$$

Remark 1. It is easy to check that (2.3) follows from the following condition

$$|\phi(r)| \le C|b(r)| \quad \text{for } |r| \ge r_0. \tag{2.3'}$$

Our main result is the following theorem.

Theorem 2.1. *Assume that (H1) and (H2) hold. Then any weak solution of (E) is an entropy solution.*

Remark 2. If b is strictly increasing, then we have from [8, Corollary 9] the following equivalence:

Theorem 2.2. *Let (H1) hold. Let $b^{-1} \in C(\mathbb{R})$. Then any weak solution of (E) is an entropy solution.*

As a direct consequence of the above theorem and [8,Corollary 10] we have the following uniqueness theorem for weak solutions.

Corollary 2.3. *Assume that (H1) and (H2) hold. Let $g_{i0} \in L^1(\Omega)$, $g_{i0} \in R(g)$ and $f_i \in L^2((0,T);H^{-1}(\Omega)) \cap L^1(Q)$. Let u_i be a weak solution of (E) with $f = f_i$ and $g_0 = g_{i0}$ for $i = 1,2$. Then*

$$\begin{aligned}&\|g(u_1(t)) - g(u_2(t))\|_{L^1(\Omega)}\\ &\quad \le \|g_{10} - g_{20}\|_{L^1(\Omega)} + \int_0^t \|f_1(s) - f_2(s)\|_{L^1(\Omega)} ds.\end{aligned}$$

We here give some examples.

Example 1. *Let $b : \mathbb{R} \to \mathbb{R}$ be continuous and nondecreasing function satisfying $b(0) =$. We consider the equation of porous medium type:*

$$(E_1) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta b(u) = f & \text{in } Q, \\ b(u) = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases}$$

Then, Theorem 2.1 with $\phi \equiv 0$ implies that any weak solution of (E) is an entropy solution. Hence, by Corollary 2.3 we have the uniqueness of weak solutions of (E_1). (See [6].)

Example 2. *Let us consider the equation*

$$(E_2) \quad \begin{cases} \frac{\partial v}{\partial t} - \Delta w + div(\psi(w)v) = f, \quad w \in \beta(v) & \text{in } Q, \\ w = 0 & \text{on } (0,T) \times \partial\Omega, \\ v(0,\cdot) = v_0 & \text{in } \Omega. \end{cases}$$

where β is a (possibly multi-valued) maximal monotone operator with $\beta(0) \ni 0$ and $\psi \in C(\mathbb{R}, \mathbb{R}^N)$. We set $v + w = u$, $(I + \beta)^{-1} = g$ and $(I + \beta^{-1} = b$, then $v = g(u)$, $w = b(u)$

and $\phi(u) = \psi(b(u))g(u)$. *Hence,* (E_2) *becomes*

$$(E_2')\quad \begin{cases} \frac{\partial g(u)}{\partial t} - \Delta b(u) + div\,\phi(u) = f & in \quad Q, \\ b(u) = 0 & on \quad (o,T)\times\partial\Omega, \\ g(u(0,\cdot)) = v_0 & in \quad \Omega. \end{cases}$$

If beta and psi satisfy in addition that

$$|\psi(b(r))g(r)| \le Cb(r)^2$$

for all sufficiently large $|r|$, *then any weak solution of* (E_2') *is an entropy solution.*

Example 3. *Let us consider the equation*

$$(E_3)\quad \begin{cases} \frac{\partial v}{\partial t} - \Delta\beta(v) + e\cdot\nabla\chi = f, \quad \chi\in\gamma(v) & in \quad Q \\ v = 0 & on \quad (0,T)\times\partial\Omega, \\ v(0,\cdot) = v_0 & in \quad \Omega. \end{cases}$$

where β *is a continuous and nondecreasing function with* $\beta(0) = 0$, γ *is (possibly multi-valued) maximal monotone operator with* $\gamma(0) \ni 0$ *and* $e \in \mathbb{R}^N$. *Then, if we set* $u = v + \chi, g = (I+\gamma^{-1}, b = \beta\circ g$ *and* $\phi = e(I-g)$,*then* (E_3) *converts into*

$$(E_3')\quad \begin{cases} \frac{\partial g(u)}{\partial t} - \Delta b(u) + div\,\phi(u) = f & in \quad Q, \\ b(u) = 0 & on \quad (0,T)\times\partial\Omega, \\ g(u(0,\cdot)) = v_0. & in \quad \Omega. \end{cases}$$

If b *is in addition strictly increasing , then Theorem 2.2 assures that any weak solution of* (E_3') *is an entropy solution.*

Example 4. *Let us finally consider the following equation of dam problems (see [9]):*

$$(E_4)\quad \begin{cases} \frac{\partial(\chi+v)}{\partial t} = \Delta v + div(\chi e), \quad \chi\in\gamma(v) \quad in \quad Q \\ \chi(0,\cdot) + v(0,\cdot) = \chi_0 + v_0 \end{cases}$$

where γ *is a maximal graph in* $\mathbb{R}$. *If we set* $u = \chi + v$, $g(u) = u$, $b(u) = (I+\gamma)^{-1}(u)$ *and* $\phi(u) = e(u-v) = e(g(u) - b(u))$, *then* $v = (I+\gamma)^{-1} = b(u)$ *and* (E_4) *becomes*

$$\frac{\partial g(u)}{\partial t} = \Delta b(u) + div\,\phi(u).$$

To check Condition (H2) we set

$$\phi^{(1)}(r) = -er \qquad and \qquad \phi^{(2)}(r) = e$$

Since $b(u) = 0$ *implies* $\gamma(0) \ni u$, *(2.1) holds true whenever* $\gamma(0) = \{0\}$. *(2.3) is satiafied wnenever* $\underline{\lim}_{|u|\to\infty}|b(u)|^2/|u| > 0$ *(see (2.3′)).*

3. Sketch of Proof of the main theorem

Let us mention briefly the main ingredients of the proof. For the complete proof we refer to [13]. We begin with the following lemma which is an evolutional version of [8,Lemma 2].

Lemma 3.1. *Let $e \in \mathbb{R}^N$, $F \in L^2(Q)^N$ and $G_0, G_1 \in L^1(Q)$. Let u be a measurable function on Q such that $g(u) \in L^1(Q)$, $\partial g(u)/\partial t \in L^2(0,T;H^{-1}(\Omega))$ and $g(u(0,x)) = g_0(x)$ a.e. x in Ω. Suppose that there are $m_0, m_1 \in \mathbb{R}$ with $m_1 < m_0$ such that the following inequality holds with s replaced by m_i and G replaced by G_i, $i = 0,1$:*

$$\int_Q \{((g(u) - g(s))^+(-\xi_t + e \cdot \nabla\xi) + F \cdot \nabla\xi + G\xi\}dtdx \qquad (3.1)$$
$$\leq \int_\Omega (g_0 - g(s))^+\xi(0)dx$$

for any $\xi \in \mathcal{D}_T^+$, where

$$\mathcal{D}_T^+ = C_0^\infty([0,T) \times \Omega)^+ \quad or \quad \mathcal{D}_T^+ = C_0^\infty([0,T) \times \bar{\Omega})^+.$$

Then, (3.1) is also valid with $G = G_1\chi + G_0(1-\chi)$ for any $s \in [m_1, m_0]$, for any $\xi \in \mathcal{D}_T^+$, and for some $\chi \in H((g(u) - g(m_1))^+ + g(m_1) - g(s) - (g(u) - g(m_0))^+)$.

Lemma 3.2. *Any weak solution of (E) is a pre-entropy solution, that is, (1.1) and (1.2) hold for any $(s,\xi) \in \mathbb{R} \times C_0^\infty([0,T) \times \Omega)^+$.*

Proof. Let $s \in \mathbb{R}$ and $i = 0,1$. If $b(s) \notin E$, then it follows from [8, Lemma 5] that (1.1) and (1.2) hold for any $\xi \in C_0^\infty((0,T) \times \Omega)$. Take any $s \in \mathbb{R}$ with $b(s) \in E$ and let $b^{-1}(b(s)) = [m_1, m_0]$. If m_i is finite, there exists a sequence $\{s_n^i\}$ such that $s_n^0 > m_0$, $s_n^1 < m_1$, $b(s_n^i) \notin E$, $s_n^i \to m_i$ and $H_0(u - s_n^i) \to H_i(u - m_i)$ a.e. in Ω. Then, passing $n \to \infty$ in (1.1) with $s = s_n^i$ yields

$$\int_Q H_i(u - m_i)\{\nabla b(u) \cdot \nabla\xi - (\phi(u) - \phi(m_i)) \cdot \nabla\xi \qquad (3.2)$$
$$- (g(u) - g(m_i))\xi_t - f\xi\}dtdx \leq \int_\Omega (g_0 - g(m_i))^+\xi(0)dx$$

for any $\xi \in C_0^\infty([0,T) \times \Omega)^+$. If $m_0 = \infty$ and $m_1 = -\infty$, then $H_0(u - m_0) \equiv 0$ and $H_1(u - m_1) \equiv 1$; hence (3.3) is still valid because u is a weak solution. For the letter use we here remark that (3.3) also holds for any $\xi \in C_0^\infty([0,T) \times \bar{\Omega})^+$ provided $m_i \geq 0$. We apply Lemma 3.1 with

$$\begin{aligned}
e &= -\phi^{(2)}(b(m_0)) = -\phi^{(2)}(b(m_1)) = -\phi^{(2)}(b(s)), \\
F &= H_0(u - s)\{\nabla b(u) + \phi^{(1)}(s)) - \phi^{(1)}(b(u)) \\
&\quad + (\phi^{(2)}(b(s)) - \phi^{(2)}(b(u)))g(u)\}, \\
G_i &= -H_i(u - m_i)f
\end{aligned}$$

$$\int_Q H_0(u-s)\{\nabla b(u)\cdot\nabla\xi - (\phi(u)-\phi(s))\cdot\nabla\xi \tag{3.3}$$
$$- (g(u)-g(s))\xi_t - f\chi_1\xi\}dtdx \le \int_\Omega (g_0 - g(s))^+\xi(0)dx,$$

where $\chi_1 = H_0(u-m_0)(1-\chi) + H_1(u-m_1)\chi$ for some $\chi \in H((g(u)-g(m_1))^+ + g(m_1) - g(s) - (g(u)-g(m_0))^+)$.

Now, for any $s \in [m_1, m_0)$ there exists a sequence $\{s_n\}$ such that $s < s_n < m_0$ and $s_n \to s$. Then, passing $n \to \infty$ in (3.4) with $s = s_n$ yields

$$\int_Q H_0(u-s)\{\nabla b(u)\cdot\nabla\xi - (\phi(u)-\phi(s))\cdot\nabla\xi$$
$$- (g(u)-g(s))\xi_t - f\xi\}dtdx \le \int_\Omega (g_0 - g(s))^+\xi(0)dx.$$

Thus, (1.1) holds for any $(s,\xi) \in \mathbb{R}\times C_0^\infty([0,T)\times\Omega)^+$.

Similarly we can prove that (1.2) holds for any such (s,ξ). □

The next lemma is crucial in our argument.

Lemma 3.3. *Any pre-entropy solution of (E) satisfies*

$$\int_Q \tilde{H}_+(u)\{(\nabla b(u)-\phi(u))\cdot\nabla\xi - g(u)\xi_t - f\xi\}dtdx \le \int_\Omega g_0^+\xi(0)dx \tag{3.4}$$

and

$$\int_Q \tilde{H}_-(-u)\{(\nabla b(u)-\phi(u))\cdot\nabla\xi - g(u)\xi_t - f\xi\}dtdx \ge -\int_\Omega g_0^-\xi(0)dx \tag{3.5}$$

for any $\xi \in C_0^\infty([0,T)\times\bar{\Omega})^+$ and for some $\tilde{H}_\pm(\pm u) \in H(\pm u)$.

Proof. We assume that $u = u(t,x)$ is a pre-entropy solution of (E). Let $\zeta = \zeta(t,x,s,y)$ be a smooth function in $\mathbb{R}^{2N+2}$ such that

$$\begin{aligned} &(s,y) \mapsto \zeta(t,x,s,y) \in C_0^\infty((0,T)\times\bar{\Omega})^+ \quad \text{for each} \quad (t,x)\in Q,\\ &(t,x) \mapsto \zeta(t,x,s,y) \in C_0^\infty([0,T)\times\Omega)^+ \quad \text{for each} \quad (s,y)\in Q. \end{aligned} \tag{3.6}$$

Let $\eta = \eta(s,y) \in C_0^\infty(Q)^+$. Then, from the definition of pre-entropy solution we have

$$\int_{Q\times Q} H_0(u)\{(\nabla b(u)-\phi(u))\cdot(\nabla_x\zeta + \nabla_y\zeta) \tag{3.7}$$
$$- g(u)(\zeta_t + \zeta_s) - f\zeta\}\eta dtdxdsdy$$
$$\le \int_{(\{0\}\times\Omega)\times Q} g_0^+\zeta\eta dxdsdy$$
$$- \int_{Q\times Q}\{(\nabla_x b(u^+) - \phi(u^+))\cdot\nabla_y\eta - g(u^+)\eta_s\}\zeta dtdxdsdy.$$

Now let $\xi \in C_0^\infty([0,T)\times C)^+$, where C is a bounded open cylinder in $\mathbb{R}^N$ for which either $C \subset \Omega$ or $C \cap \partial\Omega$ is a part of the graph of a Lipschitz continuous function. Then there exists a sequence of mollifiers σ_ℓ defined on $\mathbb{R}$ with $\operatorname{supp}\sigma_\ell \subset (-2/\ell, 0)$ and there exists a sequence of mollifires ρ_n defined on $\mathbb{R}^N$ such that

$$x \mapsto \rho_n(x-y) \in C_0^\infty(\Omega) \quad \text{for each} \quad y \in C \cap \Omega.$$

The function $\zeta^{(n,\ell)}$ defined by

$$\zeta^{(n,\ell)}(t,x,s,y) = \xi(t,x)\rho_n(t-s)\sigma_\ell(t-s)$$

satisfies (3.7) and

$$\nabla_x \zeta^{(n,\ell)} + \nabla_y \zeta^{(n,\ell)} = \rho_n \sigma_\ell \nabla_x \xi, \quad \zeta_t^{(n,\ell)} + \zeta_s^{(n,\ell)} = \xi_t \rho_n \sigma_\ell.$$

Using $\zeta^{(n,\ell)}$ as a test function ζ in (3.8) and passing to the limit with $n, \ell \to \infty$,

$$\begin{aligned} &\int_Q \tilde{H}_+(u)\{(\nabla b(u) - \phi(u)) \cdot \nabla \xi - g(u)\xi_s - f\xi\}\eta dsdy \\ &\le - \int_Q \{(\nabla b(u^+) - \phi(u^+)) \cdot \nabla \eta - g(u^+)\eta_s\}\xi dsdy \end{aligned} \tag{3.8}$$

for some $\tilde{H}_+(u) \in H(u)$.

On the other hand, from the definition of pre-entropy solution we have that $\operatorname{div}\{\nabla b(u^+) - \phi(u^+)\} - (\partial/\partial t)g(u^+)$ becomes a Radon measure on Q. Therefore, passing to the limit in (3.9) with $\eta \to 1$ on Q, we obtain

$$\begin{aligned} &\int_Q \tilde{H}_+(u)\{(\nabla b(u) - \phi(u)) \cdot \nabla \xi - g(u)\xi_t - f\xi\}dtdx \\ &\le \int_Q \{\operatorname{div}((\nabla b(u^+) - \phi(u^+))\xi) - \frac{\partial}{\partial t}(g(u^+)\xi)\}dtdx \\ &= \int_\Omega g_0^+ \xi(0) dx + \int_0^T \int_{\partial(\Omega \cap C)} (\nabla b(u^+) - \phi(u^+)) \cdot \xi \nu d\mathcal{H}^{N-1}. \end{aligned} \tag{3.9}$$

Here we used the result in [10, Theorem 2.2], and ν is the outward unit normal to $\partial(\Omega \cap C)$ and $\mathcal{H}^{N-1}$ is the usual $(N-1)$-dimensional Hausdorff measure (e.g. see [11]). We recall from [10] that the second integral on the right hand of (3.10) is defined by

$$\begin{aligned} &\int_0^T \int_{\partial(\Omega\cap C)} (\nabla b(u^+) - \phi(u^+)) \cdot \xi \nu dt d\mathcal{H}^{N-1} \\ &= \operatorname{ess.} \lim_{\tau \to +0} \int_0^T \int_{\partial(\Omega \cap C)} (\nabla b(u^+ \circ \Psi_\tau) - \phi(u^+ \circ \Psi_\tau))\xi \cdot (\nu_\tau \circ \Psi_\tau) J\Psi_\tau dt d\mathcal{H}^{N-1}, \end{aligned}$$

where $\Psi_\tau(\cdot) = \Psi(\cdot, \tau)$ with a Lipschitz deformation $\Psi : \partial(\Omega \cap C) \times [0,1] \to \overline{\Omega \cap C}$, ν_τ is the outward unit normal to $\Psi_\tau(\partial(\Omega \cap C))$ and $J\Psi_\tau$ denotes the Jacobian of Ψ_τ.

Choose a deformation Ψ of $\partial(\Omega \cap C)$ and $p \in \mathbb{R}^N$ with the following properties: If $\tau > 0$ is sufficiently small, then $\Psi_\tau(x) = x + \tau p$, $\nu_\tau(\Psi_\tau(x)) = \nu(x)$ and $p \cdot \nu(x) < 0$ for each $x \in (C \cap \partial\Omega) \setminus N_\tau$ for some $\mathcal{H}^{N-1}$- measurable set N_τ whose $\mathcal{H}^{N-1}$-measure is less than τ. Since $b(u^+) \in L^2(0,T; H_0^1(\Omega)^+)$, it follows that

$$\limsup_{\tau \to +0} \int_0^T \int_{\partial\Omega \cap C} \nabla b(u^+ \circ \Psi_\tau) \cdot p dt d\mathcal{H}^{N-1} \ge 0,$$

which deduces

$$\int_0^T \int_{\partial(\Omega\cap C)} (\nabla b(u^+) - \phi(u^+)) \cdot \nu \xi dt d\mathcal{H}^{N-1} \tag{3.10}$$
$$\leq \limsup_{\tau \to +0} \int_0^T \int_{\partial\Omega\cap C} -\phi(u^+ \circ \Psi_\tau) \cdot \nu \xi J \Psi_\tau dt d\mathcal{H}^{N-1}.$$

Now by virtue of (H2) we have that for any $\varepsilon > 0$ there exist a constant $C_\varepsilon \geq 0$ and a continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$, which is independent of ε, such that

$$|\phi(r)| \leq \rho(\varepsilon) + C_\varepsilon b(r)^2 \quad \text{for all} \quad r \in \mathbb{R}.$$

Noting that $b(u^+)^2 \in L^1(0,T; W_0^{1,1}(\Omega))$, we see that the right hand of (3.11) can be estimated from above by

$$C\rho(\varepsilon) + C_\varepsilon \int_0^T \int_{\partial\Omega} b(u^+)^2 dt d\mathcal{H}^{N-1} = C\rho(\varepsilon).$$

Letting $\varepsilon \to 0$, we obtain from (3.9),(3.10) and (3.11) that for any $\xi \in C_0^\infty([0,T) \times C)^+$,

$$\int_Q \tilde{H}_+\{(\nabla b(u) - \phi(u)) \cdot \nabla \xi - g(u)\xi_t f\xi\} dt dx \leq \int_\Omega g_0^+ \xi(0) dx. \tag{3.11}$$

Now let $\{C_i\}_{i=0}^k$ be bounded open cylinders such that $\Omega \subset \bigcup_{i=0}^k C_i$, $C_0 \subset \Omega$ and for $i \geq 1$, $C_i \cap \partial\Omega$ is a part of the graph of a Lipschitz function. Let $\{\varphi_i\}_{i=0}^k$ be a partition of unity subordinate to the covering $\{C_i\}$. Let $\xi \in C_0^\infty([0,T) \times \bar{\Omega})^+$ and let $\xi_i = \xi\varphi_i$. Since $\xi_i \in C_0^\infty([0,T) \times C_i)^+$, it then holds that (3.12) is valid for ξ_i instead of ξ. By adding the resultant inequalities with respect to i we obtain the desired inequality (3.5).

Similarly we obtain (3.6). □

We are now in the position to prove our main theorem.

Proof of Theorem 2.1. By virtue of [8, Lemma 5] and Lemma 3.2 above it sufficies to prove (1.1) and (1.2) for every $(s,\xi) \in \mathbb{R}^+ \times C_0^\infty([0,T) \times \bar{\Omega})^+$ *such that* $b(s) \in E$, *where* $E = \{r \in R(b) : b^{-1}(r) \textit{ is not a singleton}\}$. *From now on we fix such* s *and* ξ *and let* $b^{-1} \circ b(s) = [m_1, m_0]$. *We shall follow the same argument as in the proof of Lemma 3.2. As was remarked there, (3.3) also holds for every* $\xi \in C_0^\infty([0,T) \times \bar{\Omega})^+$ *provided* $m_i \geq 0$. *However, we always have that* $m_0 \geq 0$. *On the other hand, by taking account of Lemma 3.3 we have*

$$\int_Q \tilde{H}_+(u - m_1^+)\{\nabla b(u) \cdot \nabla \xi - (\phi(u) - \phi(m_1^+)) \cdot \nabla \xi$$
$$- (g(u) - g(m_1^+))\xi_t - f\xi\} dt dx$$
$$\leq \int_\Omega (g_0 - g(m_1^+))^+ dx$$

for some $\tilde{H}_+(u - m_1^+) \in H(u - m_1^+)$.

Thanks to these inequalities, the proof is the same as that of Lemma 3.2 except that m_1 *and* H_1 *are replaced by* m_1^+ *and* $\tilde{H}_+$, *respectively. Consequently, we obtain (1.1) for any* $(s,\xi) \in \mathbb{R}^+ \times C_0^\infty([0,T) \times \bar{\Omega})^+$.

Similarly, (1.2) can be proved for any such s and ξ. Thus the proof of the main result is complete.

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